

The Functor of Points of a Scheme

Mumford, *Red Book*, Ch. II, §6

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The Functor of Points: Intuition

Points of schemes are weird. Non-closed points are strange enough, but worse: the fibre product of two schemes $X \times_S Y$ does NOT in general map injectively into the set-theoretic product $X \times Y$:

Example 1

Consider $\mathrm{Spec}(\mathbb{C})$ as a scheme over $\mathbb{R}$ via $\mathbb{R} \hookrightarrow \mathbb{C}$. Then

$$\mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \cong \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathrm{Spec}(\mathbb{C} \times \mathbb{C}) \cong \mathrm{Spec}(\mathbb{C}) \amalg \mathrm{Spec}(\mathbb{C}),$$

since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2 + 1) \cong \mathbb{C} \times \mathbb{C}$ (a 4-dimensional $\mathbb{R}$ -algebra, not a field). So the fibre product has two points, while the set-theoretic product of two one-point spaces has one.

As Mumford notes, this is what prevents the underlying point set of a group variety from being an abstract group (in a natural way) [▶ why?](#). The resolution: there are really two notions of “point” in the language of schemes. (We will come back and identify the two points of this example later.)

The Functor of Points: Issues

Idea: in many categories C , a single “test object” z recovers the underlying point set via $X \mapsto \text{hom}_C(z, X)$:

- $C =$ differentiable manifolds, $z =$ one-point manifold (terminal): $\text{hom}_C(z, X) \cong X$.
- $C =$ groups, $z = \mathbb{Z}$ (free on one generator, not terminal): $\text{hom}_C(\mathbb{Z}, G) \cong G$.
- $C =$ rings with unity, $z = \mathbb{Z}[x]$ (likewise free): $\text{hom}_C(\mathbb{Z}[x], R) \cong R$ as sets.

But this can also fail to be faithful, i.e. a morphism $f : X_1 \rightarrow X_2$ need not be determined by the induced map of sets $\text{hom}_C(z, X_1) \rightarrow \text{hom}_C(z, X_2)$:

- $C = (\text{Hot})$, CW-complexes with homotopy classes of maps, $z =$ point: $\text{hom}(z, X) = \pi_0(X)$. Not faithful in general!
- $C =$ schemes: no scheme plays the role of $\mathbb{Z}[x]$, so the only candidate is the terminal $z = \text{Spec}(\mathbb{Z})$, and $\text{hom}(\text{Spec}(\mathbb{Z}), X)$ is absurdly small, e.g. $\text{hom}(\text{Spec}(\mathbb{Z}), \text{Spec}(\mathbb{F}_{p^2})) = \emptyset$, though $\text{id} \neq \text{Frob}$. Not faithful!

So for schemes, no one object z can work. Grothendieck's idea was to use all of them at once:

The Functor of Points: Definition

Definition 2

Let C be a (locally small) category and X an object of C . The functor of points of X is the contravariant functor $h_X : C^{\text{opp}} \rightarrow \text{Set}$ given by

$$\begin{aligned} h_X(Y) &= \text{hom}_C(Y, X) && \text{on objects,} \\ h_X(f : Y \rightarrow Z) &= (-) \circ f : \text{hom}_C(Z, X) \rightarrow \text{hom}_C(Y, X) && \text{on morphisms.} \end{aligned}$$

Question 3

Why do we need locally small? Why do we need contravarianness?

Exercise 4

Show this is actually a functor.

The Functor of Points: Definition

Proposition 5 (Yoneda; Mumford, Prop. 1)

h is fully faithful, i.e. if Y, Z are objects in C , then under h

$$\mathrm{hom}_C(Y, Z) \cong \mathrm{hom}_{\mathrm{Funct}}(h_Y, h_Z).$$

Proof sketch.

Postcomposition with $f \in \mathrm{hom}_C(Y, Z)$ gives $h_f : h_Y \rightarrow h_Z$. Conversely, given $\Phi : h_Y \rightarrow h_Z$, evaluate at the “universal element”: $f := \Phi_Y(\mathrm{id}_Y)$. Naturality applied to any $g : W \rightarrow Y$ forces $\Phi_W(g) = f \circ g$, so $\Phi = h_f$. ▢

▸ full proof

Heuristic conclusion (Mumford): an object X of C can be identified with its functor h_X , which is “basically just a structured set.” (Mumford’s own proof, in its entirety: “Easy.”)

Scheme Valued Points

Definition 6

Let X and K be schemes. A K -valued point of X is a morphism $f : K \rightarrow X$. If $K = \operatorname{Spec}(R)$, we call this an R -valued point of X . If X and K are schemes over a base S (via $p_X : X \rightarrow S$, $p_K : K \rightarrow S$), then f is a K -valued point of X/S if $p_X \circ f = p_K$. The set of all R -valued points of X (or of X/S) is denoted $X(R)$.

Remark 7

Points in this sense ARE compatible with products: if K, X, Y are schemes over S , then

$$(X \times_S Y)(K) \cong X(K) \times Y(K) \quad (\text{as sets, over } S).$$

This is just the universal property of the fibre product (why?). In particular, K -valued points of a group scheme really do form a group (why?). What else transfers in general (i.e. when we have finite products)?

Closing the Loop: What ARE Those Two Points?

Back to $\mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$, now with the right language. Compute $\mathbb{C}$ -valued points (over $\mathbb{R}$): the fibre product's $\mathbb{C}$ -points are $\mathrm{Spec}(\mathbb{C})(\mathbb{C}) \times \mathrm{Spec}(\mathbb{C})(\mathbb{C})$, and

$$\mathrm{Spec}(\mathbb{C})(\mathbb{C}) = \mathrm{Hom}_{\mathbb{R}\text{-alg}}(\mathbb{C}, \mathbb{C}) = \{\mathrm{id}, \sigma\} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

So the fibre product has four $\mathbb{C}$ -points on its two topological points. Which two? Each factor of

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}, \quad a \otimes b \mapsto (ab, a\sigma(b))$$

is an $\mathbb{R}$ -algebra map $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{C}$: multiplication $a \otimes b \mapsto ab$, and its twist $a \otimes b \mapsto a\sigma(b)$. Their kernels are the two points, i.e. the graphs of id and σ .

[▶ details](#)

Remark 8 (Moral)

The “mystery point” is complex conjugation. The underlying space of $\mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ is secretly the Galois group: the functor of points sees arithmetic that the topology hides.

Why Any of This Even Matters

R -valued points generalize solutions of Diophantine equations. Let $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$ and

$$X = \operatorname{Spec}(\mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m)).$$

Claim 9

An R -valued point of X is the “same thing” as an n -tuple $a_1, \dots, a_n \in R$ with

$$f_1(a_1, \dots, a_n) = \dots = f_m(a_1, \dots, a_n) = 0.$$

Indeed, a morphism $g : \operatorname{Spec}(R) \rightarrow X$ is determined by the tuple $a_i = g^*(x_i)$, and the tuples that occur are exactly those for which $h \mapsto h(a_1, \dots, a_n)$ is a well-defined homomorphism $\mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m) \rightarrow R$ i.e. solutions of $f_1, \dots, f_m$.

E.g. $X = \operatorname{Spec}(\mathbb{Z}[x, y, z]/(x^n + y^n - z^n))$: then $X(\mathbb{Z})$ is the set of integer solutions of Fermat's equation.

Representability

It turns out a scheme is determined by even less information than testing on all S -points: define the covariant functor $h_X^0 : (\text{Rings}) \rightarrow \text{Set}$ by $h_X^0(R) = h_X(\text{Spec}(R)) = \text{hom}(\text{Spec}(R), X)$.

Proposition 10 (Mumford, Prop. 2)

For any two schemes X, Y ,

$$\text{hom}(X, Y) \cong \text{hom}(h_X^0, h_Y^0).$$

Hence h^0 is a fully faithful functor from the category of schemes to $\text{Func}(\text{Rings}, \text{Set})$.

Idea in one line: an affine cover of X turns a natural transformation $F : h_X^0 \rightarrow h_Y^0$ into gluing data for an honest morphism $X \rightarrow Y$.

► proof

Grothendieck's Existence Problem

Why not go further and define schemes as suitable functors $F : (\text{Rings}) \rightarrow \text{Set}$? The problem: find “natural” conditions on F ensuring $F \cong h_X^0$ for some scheme X .

One necessary condition (Grothendieck): compatibility with faithfully flat descent. If $A \rightarrow B$ is faithfully flat, then

$$F(A) \rightarrow F(B) \rightrightarrows F(B \otimes_A B)$$

is exact/an equalizer diagram (i.e. F is a sheaf for the fpqc topology)

Remark 11

In Mumford's day this was an open-ended “problem”; today it has a definite answer: F is (representable by) a scheme iff F is a Zariski sheaf admitting an open cover by affine representable subfunctors. Thus, now we really can define a scheme via a functor. Also, note that weakening “open cover by affines” to étale/fppf covers leads to algebraic spaces and stacks.

Aside: Technical Issues

Grothendieck's existence problem matters because one may start with a scheme, pass to its functor of points, perform some natural operation and not know whether the result is a scheme!

Yoneda transports limits for free ($h_{X \times_S Y} = h_X \times_{h_S} h_Y$), but representability is not closed under most other operations:

- **Quotients:** a finite group acting on a quasi-projective scheme over a ring has a scheme quotient (what is really needed: every orbit lies in an affine open – freeness only makes the quotient map étale); drop quasi-projectivity and this fails (Hironaka: smooth proper 3-fold, free $\mathbb{Z}/2$ -action, quotient only an algebraic space).
- **Colimits:** nilpotents of order $\leq n$ are representable by $\text{Spec } \mathbb{Z}[x]/(x^n)$, but all nilpotents form only a formal scheme $(\widehat{\mathbb{G}}_a)$ i.e. truncations are representable, their colimit is not.
- **Sheaf quotients:** $h_X / \sim$ for an étale equivalence relation need not be a scheme.

Underlying obstruction: scheme-hood requires enough affine opens, a condition invisible to functor-level constructions.

Field Valued Points

Consider the case where $R = k$ is a field. We know $\mathrm{Spec}(k)$ has one point, so any map $f : \mathrm{Spec}(k) \rightarrow X$ has one point as its image, say $x \in X$. The induced map on stalks $f^* : \mathcal{O}_{X,x} \rightarrow k$ must be local, so $\mathfrak{m}_x$ is sent to zero, i.e. we get a well-defined embedding $\kappa(x) \hookrightarrow k$. Conversely, given $x \in X$ and an inclusion $\kappa(x) \subset k$, we get a k -valued point with image x by defining f^* as $\Gamma(U, \mathcal{O}_X) \rightarrow \mathcal{O}_{X,x} \rightarrow \kappa(x) \rightarrow k$ for all open $U \ni x$. Hence:

$$\{k\text{-valued points of } X\} \cong \{\text{pairs } (x, \iota) : x \in X, \iota : \kappa(x) \hookrightarrow k\}.$$

Example 12

For each $x \in X$ there is a canonical morphism $i_x : \mathrm{Spec}(\kappa(x)) \rightarrow X$ with image x . If k is algebraically closed and X is of finite type over k , then

$$X(k) \cong \{x \in X : k \xrightarrow{\sim} \kappa(x)\} = \{\text{closed points of } X\},$$

recovering the underlying set of the classical variety. These are geometric points (over non-closed k : only rational ones). [▶ why?](#)

Local Ring Valued Points

Now let R be any local ring with maximal ideal $\mathfrak{m}$. Then $\mathrm{Spec}(R)$ has a unique closed point $[\mathfrak{m}]$, and if $f : \mathrm{Spec}(R) \rightarrow X$ is a morphism, any open set containing $f([\mathfrak{m}])$ contains the whole image of f (why?). In particular the image always lies in one affine piece of X .

Proposition 13 (Mumford, Prop. 3)

Let $x \in X$. There is a bijection

$$\{R\text{-valued points } f \text{ of } X \text{ with } f([\mathfrak{m}]) = x\} \cong \{\text{local homomorphisms } q : \mathcal{O}_{X,x} \rightarrow R\}.$$

Idea in one line: reduce to $X = \mathrm{Spec}(A)$ affine (image lies in one affine piece!) and match both sides with $\{\varphi : A \rightarrow R \text{ with } \varphi^{-1}(\mathfrak{m}) = \mathfrak{p}\}$. ▶ proof

Why bother with rings beyond fields? Because X and X_{red} have the same K -points for every field K : fields cannot see nilpotents, but local rings can. E.g. the fat point $X = \mathrm{Spec} k[x]/(x^2)$ has $X(K) = \{0\}$ for every field K , but $X(k[\varepsilon]) \cong k$.

Application: Tangent Vectors

Let $k[\varepsilon] := k[x]/(x^2)$, local with maximal ideal (ε) , so Prop. 3 applies. The k -algebra map $k[\varepsilon] \rightarrow k$, $\varepsilon \mapsto 0$, induces a reduction map $X(k[\varepsilon]) \rightarrow X(k)$; write $X(k[\varepsilon])_x$ for the fibre over x .

Example 14 (Tangent vectors)

Let X be a scheme over k and $x \in X(k)$ a k -point. Then

$$X(k[\varepsilon])_x \cong \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k) = T_x X.$$

Sketch.

By Prop. 3 such an f is a local k -algebra map $\varphi : \mathcal{O}_{X,x} \rightarrow k[\varepsilon]$. Uniquely $\varphi(a) = c(a) + \varepsilon D(a)$; reducing mod ε shows c is the residue map, so $c(a) = a(x)$ — this is exactly the lifting condition. Multiplicativity with $\varepsilon^2 = 0$ is then the Leibniz rule $D(ab) = a(x)D(b) + b(x)D(a)$, so D kills constants and $\mathfrak{m}_x^2$: a functional on $\mathfrak{m}_x/\mathfrak{m}_x^2$. [details](#) □

Moral: infinitesimals are built into the functor of points!

Practice Problems

Compute the functor of points $R \mapsto X(R)$ for the following schemes X :

- $\mathbb{A}_{\mathbb{Z}}^n = \operatorname{Spec}(\mathbb{Z}[x_1, \dots, x_n])$
- $\mathbb{G}_m = \operatorname{Spec}(\mathbb{Z}[t, t^{-1}])$
- $\mu_n = \operatorname{Spec}(\mathbb{Z}[t]/(t^n - 1))$
- $\operatorname{Spec}(\mathbb{Z}[x]/(x^2))$
- $\operatorname{GL}_2 = \operatorname{Spec}(\mathbb{Z}[a, b, c, d, t]/((ad - bc)t - 1))$
- (Harder) $\mathbb{P}_{\mathbb{Z}}^n$... what goes wrong with the naive guess ... what is it really? (Hint: consider line bundles)

► solutions

Separated Schemes

We have reached our final topic: the Hausdorff condition. The Zariski topology is essentially never Hausdorff (exercise: for $X = \operatorname{Spec}(A)$, show $\text{Hausdorff} \iff \text{T1} \iff \dim A = 0$, and that such X are exactly the profinite spaces so this is weaker than finite discrete!), so we need the right replacement. The idea: probe X with a test scheme S and two S -valued points $f, g : S \rightarrow X$, and ask where they “agree.”

Definition 15

For $s \in S$, say $f(s) \equiv g(s)$ if $f \circ i_s = g \circ i_s$, where $i_s : \operatorname{Spec}(\kappa(s)) \rightarrow S$ is the canonical morphism. Equivalently: $f(s) = g(s)$ as points of X , and the two induced maps $\kappa(f(s)) \rightarrow \kappa(s)$ and $\kappa(g(s)) \rightarrow \kappa(s)$ are equal.

Warning 16 (Terminology)

Mumford writes “prescheme” for what we now call a scheme, and reserves “scheme” for what we now call a separated scheme. These slides use the modern terms.

Why $\equiv$ Is Not Equality of Points

A $\kappa(s)$ -valued point of X is a pair (x, ι) , $\iota : \kappa(x) \hookrightarrow \kappa(s)$. So $f(s) \equiv g(s)$ demands agreement in both coordinates, i.e. image point AND sheaves. The second is not redundant:

Example 17

Let $S = \operatorname{Spec}(\mathbb{C})$ and $X = \operatorname{Spec}(\mathbb{C})$, both over $\mathbb{R}$, and take

$$f = \operatorname{id}, \quad g = \operatorname{Spec}(\sigma) \quad (\sigma = \text{complex conjugation}).$$

Each of S, X has one point, so $f(s) = g(s)$ on the nose as points of X . But the induced maps $\mathbb{C} = \kappa(f(s)) \rightarrow \kappa(s) = \mathbb{C}$ are id and σ , which differ. So $\{s : f(s) \equiv g(s)\} = \emptyset$, whereas “same image point” would give all of S .

Same lesson as Example 1: the topology cannot see the Galois twist. “Agree” defined by image points alone is too coarse, and separatedness built on it would be the wrong notion; $\equiv$ is the refinement that fixes this.

Separated Schemes

Proposition 18 (Mumford, Prop. 4)

For any $f, g : S \rightarrow X$, the set $\{s \in S : f(s) \equiv g(s)\}$ is locally closed.

► proof

Definition 19

We say a scheme X is separated if for all schemes S and all S -valued points f, g of X , the set $\{s \in S : f(s) \equiv g(s)\}$ is closed.

Proposition 20 (Mumford, Prop. 5)

If X is a scheme over a ring R , then the criterion for X to be separated is satisfied for all S, f, g if it is satisfied in the single case $S = X \times_R X$, $f = p_1$, $g = p_2$.

► proof

Remark 21

In modern language: Prop. 4 says the diagonal $\Delta : X \rightarrow X \times_R X$ is always an immersion; Prop. 5 says X/R is separated iff Δ is a closed immersion, with image exactly $\{p_1 \equiv p_2\}$.

Consequences: Back to Varieties

Corollary 22 (Mumford, Cor. 1)

Let k be algebraically closed. A prevariety over k is a variety in the sense of Ch. I if and only if it is separated.

▶ proof

Proposition 23 (Local criterion, Ch. I §5)

Let X be a variety and $x \neq y \in X$. Then there is no local ring $\mathcal{O} \subset k(X)$ dominating both $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$.

▶ proof

The proof of the local criterion is a genuine payoff of local-ring-valued points: the hypothetical $\mathcal{O}$ manufactures two $\mathcal{O}$ -valued points agreeing at the generic point but not at the closed point.

Separated Schemes

To end, let us show the power of our new definition by answering a natural question: when is the intersection of two open affines actually affine again?

Proposition 24 (Mumford, Prop. 6)

If X is a separated scheme and $U, V \subset X$ are open affine sets, then $U \cap V$ is affine and the canonical homomorphism

$$\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$$

is surjective.

► proof

Idea in one line: Δ is a closed immersion, and $U \cap V = \Delta^{-1}(U \times V)$ with $U \times V$ affine in $X \times X$.

(Note: products and tensors are absolute, over $\mathbb{Z}$ which is no loss, since X separated over any affine base is separated over $\mathbb{Z}$.)

Practice Problems

Question 25

Is the fibre product of separated schemes separated?

Question 26

What is an example of a separated scheme? What is an example of a scheme that is NOT separated?

(Answers on the next slide but let's discuss first.)

Answers

Q1. Yes: if X, Y are separated over S , the diagonal of $X \times_S Y$ is obtained from Δ_X and Δ_Y by base change and composition, and closed immersions are stable under both.

Q2. Every affine scheme is separated: Δ corresponds to the surjection $A \otimes_{\mathbb{Z}} A \twoheadrightarrow A$. The classic non-example is the line with doubled origin, $X = \mathbb{A}^1 \sqcup_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$:



The two chart inclusions $f, g : \mathbb{A}^1 \rightrightarrows X$ satisfy $\{s : f(s) \equiv g(s)\} = \mathbb{A}^1 \setminus \{0\}$: open and dense, not closed so X is not separated. (It is locally closed, consistent with Prop. 4!)

Appendix: Full Proofs & Solutions

Every result above has its complete proof here; use the [◀ back](#) buttons (or the navigation arrows, bottom right) to jump back to the statement.

Aside: Why the Point Set Is Not a Group

Let G be a group scheme over S : multiplication is a *morphism* $m : G \times_S G \rightarrow G$. To multiply points of $|G|$ we would need $|G| \times |G| \rightarrow |G|$. What we have is

$$|G| \times |G| \xleftarrow{\pi} |G \times_S G| \xrightarrow{|m|} |G|, \quad \pi = (|p_1|, |p_2|).$$

Claim 27

π is surjective, not injective: $|G| \times |G|$ is a quotient of $|G \times_S G|$. So $|m|$ induces a map on $|G| \times |G|$ iff it is constant on the fibres of π — and it is not. (Well-definedness on a quotient, not naturality.)

Take $G = \mathbb{G}_m$ over $\mathbb{R}$, $x = V(t^2 + 1)$, $\kappa(x) = \mathbb{C}$: as a point x is “ $\pm i$ ”, with no record of which. The fibre of π over (x, x) is $\text{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ — the two points from Example 1. Since $m^*(t) = t_1 t_2$: on $t_2 = t_1$, $t_1^2 = -1$; on $t_2 = -t_1$, $-t_1^2 = 1$. So $|m|(P_+) = V(t + 1)$ but $|m|(P_-) = V(t - 1)$: “ $x \cdot x$ ” would have to be both -1 and $+1$.

► so why did nobody notice?

Aside: Why Nobody Noticed Classically

The obstruction is invisible in the classical setting. Let $k = \bar{k}$, let G be of finite type over k , and restrict to *closed* points. Then every residue field is k , so for closed points x, y

$$\kappa(x) \otimes_k \kappa(y) = k \otimes_k k = k,$$

a one-point fibre. So π is a bijection on closed points, $|m|$ descends, and the closed points of G form an honest group.

Remark 28

Two ways out, and we take the second:

- restrict to closed points over $k = \bar{k}$ (classical, but throws away arithmetic);
- keep all of G and use R -valued points — by Remark 6, $(G \times_S G)(R) \cong G(R) \times G(R)$ is a genuine bijection, so $G(R)$ is a group for every R , functorially.

The group law lives on the functor of points, not on the space.

Proof: Yoneda

Proof.

($\rightarrow$) Given $f \in \text{hom}_C(Y, Z)$, define $h_f : h_Y \rightarrow h_Z$ by postcomposition: $(h_f)_W(u) = f \circ u$ for $u : W \rightarrow Y$. Naturality: for $g : W' \rightarrow W$, both routes around the square send $u \in \text{hom}(W, Y)$ to $f \circ u \circ g$.

($\leftarrow$) Given a natural transformation $\Phi : h_Y \rightarrow h_Z$, set $f := \Phi_Y(\text{id}_Y) \in \text{hom}_C(Y, Z)$. For any $g : W \rightarrow Y$, naturality of Φ applied to the morphism g gives

$$\Phi_W(g) = \Phi_W(h_Y(g)(\text{id}_Y)) = h_Z(g)(\Phi_Y(\text{id}_Y)) = f \circ g,$$

so $\Phi = h_f$.

(Inverse) The constructions are mutually inverse: $h_f \mapsto (h_f)_Y(\text{id}_Y) = f \circ \text{id}_Y = f$, and $\Phi \mapsto h_{\Phi_Y(\text{id}_Y)} = \Phi$ by the computation above.



Aside: The Two Points, Explicitly

Write $u = i \otimes 1$ and $v = 1 \otimes i$ in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Under $\varphi(a \otimes b) = (ab, a\sigma(b))$ we get $\varphi(u) = (i, i)$, $\varphi(v) = (i, -i)$, hence $\varphi(uv) = (-1, 1)$. So the idempotents are $\frac{1}{2}(1 \mp uv) \mapsto (1, 0), (0, 1)$, and since $u - v \mapsto (0, 2i)$, $u + v \mapsto (2i, 0)$:

$$\mathfrak{p}_{\text{id}} = (i \otimes 1 - 1 \otimes i) = \ker(\text{mult}) \leftrightarrow 0 \times \mathbb{C}, \quad \mathfrak{p}_{\sigma} = (i \otimes 1 + 1 \otimes i) \leftrightarrow \mathbb{C} \times 0.$$

Warning 29 (The labelling is by a *relative* element)

A $\mathbb{C}$ -point is a pair (τ_1, τ_2) , acting by $a \otimes b \mapsto \tau_1(a)\tau_2(b)$. It kills $u - v$ iff $\tau_1 = \tau_2$, and kills $u + v$ iff $\tau_1 \neq \tau_2$. So $(\text{id}, \text{id}), (\sigma, \sigma)$ lie over $\mathfrak{p}_{\text{id}}$ and $(\text{id}, \sigma), (\sigma, \text{id})$ lie over $\mathfrak{p}_{\sigma}$: what separates the two points is $\tau_1^{-1}\tau_2$, not either τ_i alone. $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts simply transitively — a torsor, with no canonical basepoint until one factor is fixed.

In general, for L/k Galois with group G one has $L \otimes_k L \cong \prod_{g \in G} L$, the components being the *graphs* of the elements of G . Here: the diagonal, and the graph of complex conjugation.

Proof: Representability (Prop. 2)

Proof.

We must show every natural $F : h_X^0 \rightarrow h_Y^0$ comes from a unique morphism $f : X \rightarrow Y$. Choose an affine open cover $U_i \cong \operatorname{Spec}(A_i)$ of X , with inclusions $l_i : \operatorname{Spec}(A_i) \cong U_i \hookrightarrow X$. Each l_i is an A_i -valued point of X , so $F(l_i) =: f_i$ is an A_i -valued point of Y :

Diagram illustrating the relationship between variables U_i , X , and Y . A solid arrow labeled f_i points from U_i to Y . A solid arrow labeled f points from X to Y . A dashed arrow points from U_i to X .

Patching: on $U_i \cap U_j$, cover by affines $\text{Spec}(B)$; the restriction of f_i to $\text{Spec}(B)$ is F applied to the B -valued point $\text{Spec}(B) \hookrightarrow X$ (naturality of F in $A_i \rightarrow B$), likewise for f_j — so they agree, and the f_i glue to $f : X \rightarrow Y$. One checks $F = h_f^0$ by evaluating on R -valued points (which factor locally through the cover); uniqueness holds since f is determined by the $f_i = F(l_i)$. $\square$

Aside: Closed Points vs. k -Points

This example is really two statements, of very different depth.

Step 1 (formal, any field k). $\kappa(x)$ is a k -algebra, and a k -point is a pair (x, ι) with $\iota : \kappa(x) \rightarrow k$ a k -algebra map. Its image is a subring of k containing k , hence is k ; and ι is injective as $\kappa(x)$ is a field. So ι is the unique isomorphism, and $X(k) \cong \{x \in X : \kappa(x) = k\}$. No hypotheses used.

Step 2 (Nullstellensatz). $\{x : \kappa(x) = k\} = \{\text{closed points}\}$, needing X of finite type and $k = \bar{k}$. On an affine chart $\text{Spec}(A)$, $x = \mathfrak{p}$:

- $\subseteq$ $\kappa(x) = k$ forces $A/\mathfrak{p} = k$, so $\mathfrak{p}$ is maximal; X is Jacobson, so x is closed in X , not just in the chart.
- $\supseteq$ x closed makes $\kappa(x) = A/\mathfrak{m}$ a finitely generated k -algebra which is a field, hence finite over k (Zariski's lemma) — so $= k$ as $k = \bar{k}$.

Warning 30 (Both hypotheses are necessary)

$V(x^2 + 1) \in \mathbb{A}_{\mathbb{R}}^1$ is closed with $\kappa = \mathbb{C} \neq \mathbb{R}$: closed $\neq$ rational. And $\text{Spec}(\bar{\mathbb{Q}}(t))$ over $\bar{\mathbb{Q}}$ has one point, closed, with $\kappa = \bar{\mathbb{Q}}(t) \neq \bar{\mathbb{Q}}$: Zariski's lemma fails without finite generation.

Proof: Local Ring Valued Points (Prop. 3)

Proof.

First, $\mathcal{O}_{[\mathfrak{m}], \text{Spec}(R)} = R$, so from $f : \text{Spec}(R) \rightarrow X$ with $f([\mathfrak{m}]) = x$ we get a local homomorphism $\mathcal{O}_{X,x} \rightarrow R$ on stalks. For the bijection, we may replace X by an affine neighbourhood $\text{Spec}(A)$ of x (the image of f lies in one affine piece); say $x = [\mathfrak{p}]$. We have a commutative triangle of maps of sets:

$$\{f : \mathrm{Spec}(R) \rightarrow X, f([\mathfrak{m}]) = x\} \longrightarrow \{\text{local homs } q : A_{\mathfrak{p}} = \mathcal{O}_{X,x} \rightarrow R\},$$

both mapping onto $\{\varphi : A \rightarrow R \text{ with } \varphi^{-1}(\mathfrak{m}) = \mathfrak{p}\}$. A morphism $\text{Spec}(R) \rightarrow \text{Spec}(A)$ is a homomorphism $\varphi : A \rightarrow R$, and $f([\mathfrak{m}]) = [\mathfrak{p}]$ is exactly $\varphi^{-1}(\mathfrak{m}) = \mathfrak{p}$; such φ localize (uniquely) to local homomorphisms $A_{\mathfrak{p}} \rightarrow R$, and composing $A \rightarrow A_{\mathfrak{p}} \rightarrow R$ recovers φ . So all arrows are bijections. $\square$

Aside: The Dual Numbers Computation in Full

Fix $x \in X(k)$, so $\kappa(x) = k$ and $\mathcal{O}_{X,x} = k \oplus \mathfrak{m}_x$ as k -vector spaces; write $a(x)$ for the image of a in $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$.

- **The constant term is forced.** $\varphi : \mathcal{O}_{X,x} \rightarrow k[\varepsilon]$ is a k -algebra map, so $\varphi(a) = c(a) + \varepsilon D(a)$ uniquely. Composing with $\varepsilon \mapsto 0$ gives a local k -algebra map $\mathcal{O}_{X,x} \rightarrow k$, necessarily the residue map: $c(a) = a(x)$. Saying f lifts x is saying this composite is i_x .
- **Locality is automatic.** If $a \in \mathfrak{m}_x$ then $a(x) = 0$, so $\varphi(a) = \varepsilon D(a) \in (\varepsilon)$; and $\varphi(a) \in (\varepsilon)$ forces $a(x) = 0$. So $\varphi^{-1}((\varepsilon)) = \mathfrak{m}_x$.
- **Leibniz.** $\varphi(ab) = \varphi(a)\varphi(b) = a(x)b(x) + \varepsilon(a(x)D(b) + b(x)D(a))$, using $\varepsilon^2 = 0$. Comparing ε -coefficients gives $D(ab) = a(x)D(b) + b(x)D(a)$.
- **Hence D kills constants and $\mathfrak{m}_x^2$.** $\varphi(c) = c$ for $c \in k$ gives $D|_k = 0$; and for $a, b \in \mathfrak{m}_x$, Leibniz gives $D(ab) = 0$ since $a(x) = b(x) = 0$.

So D is determined by, and determines, a k -linear map $\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow k$. Conversely any such functional defines φ by the displayed formula, and the two constructions are inverse.

Solutions: Functors of Points

- $\mathbb{A}_{\mathbb{Z}}^n(R) = R^n$ (Mumford's Example S, $\text{hom}(\mathbb{Z}[x], R) \cong R$, iterated).
- $\mathbb{G}_m(R) = R^\times$ (a $\text{hom } \mathbb{Z}[t, t^{-1}] \rightarrow R$ is a unit of R).
- $\mu_n(R) = \{r \in R : r^n = 1\}$.
- $\text{Spec}(\mathbb{Z}[x]/(x^2))(R) = \{r \in R : r^2 = 0\}$ — square-zero elements. (Contrast the tangent vector slide: there $\text{Spec}(k[\varepsilon])$ was the test object; here $\text{Spec}(\mathbb{Z}[x]/(x^2))$ is the object being represented. Same shape of ring, opposite sides of the Hom.)
- $\text{GL}_2(R) =$ invertible 2×2 matrices over R (the variable t forces $ad - bc \in R^\times$).
- $\mathbb{P}_{\mathbb{Z}}^n(R) = \{\text{surjections } R^{n+1} \twoheadrightarrow L, L \text{ invertible}\} / \cong$. The naive guess “lines in R^{n+1} ” works for fields (and, via unimodular rows mod units, local rings) but fails in general: over an arbitrary R , an R -point may only locally look like a line — the global data is a line-bundle quotient. This is a functorial reflection of $\mathbb{P}^n$ not being affine, though it is not a proof of that; the proof is $\Gamma(\mathbb{P}_{\mathbb{Z}}^n, \mathcal{O}) = \mathbb{Z}$, so for $n \geq 1$ an affine $\mathbb{P}_{\mathbb{Z}}^n$ would have to be $\text{Spec}(\mathbb{Z})$.

Proof: Locally Closed (Prop. 4)

Proof.

Write $Z = \{s \in S : f(s) \equiv g(s)\}$, the set in question. Fix $x \in Z$, let $y = f(x) = g(x)$. Choose affine opens $U_1 = \text{Spec}(R_1) \ni y$ in X and $U_2 = \text{Spec}(R_2) \ni x$ in S with $U_2 \subset f^{-1}(U_1) \cap g^{-1}(U_1)$, inducing $f^*, g^* : R_1 \rightarrow R_2$. Let $A \subset R_2$ be the ideal generated by all $f^*(\alpha) - g^*(\alpha)$. **Claim:** $Z \cap U_2 = V(A)$. For a prime $[\mathfrak{p}_2] \in U_2$:

$$\begin{aligned} f([p_2]) \equiv g([p_2]) &\iff (f^*)^{-1}(p_2) = (g^*)^{-1}(p_2) = p_1, \text{ say, and } \bar{f}^* = \bar{g}^* : R_1/p_1 \rightarrow R_2/p_2 \\ &\iff f^*(\alpha) - g^*(\alpha) \in p_2 \quad \text{for all } \alpha \in R_1 \iff p_2 \supset A. \end{aligned}$$

So each $x \in Z$ has an open U_2 with $Z \cap U_2$ closed in U_2 ; hence Z is closed in $\bigcup_{x \in Z} U_2$, i.e. locally closed.

Proof: Reduction to the Diagonal (Prop. 5)

Proof.

Let $f, g : S \rightarrow X$ be given, S arbitrary, $\pi : X \rightarrow \text{Spec}(R)$ the structure map. Set

$$Z_1 = \{s \in S : f(s) \equiv g(s)\}, \quad Z_2 = \{s \in S : \pi(f(s)) \equiv \pi(g(s))\}.$$

Let $\mathcal{Q} \subset \mathcal{O}_S$ be the ideal generated by the functions $f^* \pi^* \alpha - g^* \pi^* \alpha$, $\alpha \in R$. Then $\mathcal{Q}$ is quasi-coherent and $1 \notin \mathcal{Q}_s \iff s \in Z_2$, so $(Z_2, \mathcal{O}_S/\mathcal{Q})$ is a closed subscheme $\mathcal{Z}$ of S ; since $Z_1 \subset Z_2$, it suffices to show Z_1 closed in $\mathcal{Z}$. By construction $(\pi f)^* = (\pi g)^* : R \rightarrow \Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$, i.e. $\pi f = \pi g$ on $\mathcal{Z}$, so f, g induce $h = (f, g) : \mathcal{Z} \rightarrow X \times_R X$. For $s \in \mathcal{Z}$ with $y = h(s)$ one checks $f(s) \equiv g(s) \iff p_1(y) \equiv p_2(y)$. Hence

$$Z_1 = h^{-1}(\{y \in X \times_R X : p_1(y) \equiv p_2(y)\}),$$

so if the set in braces is closed, so is Z_1 .

Proof: Varieties vs. Schemes (Cor. 1)

Proof.

Let $Z \subset X \times_k X$ be the set of points z with $p_1(z) \equiv p_2(z)$. If z is a closed point, then $z \in Z$ iff $p_1(z) = p_2(z)$ (residue fields are all k , so the extra condition on residue maps is automatic). By Prop. 4, Z is locally closed, and a locally closed set is closed iff its intersection with the closed points is closed in the induced topology. That step needs the closed points to be very dense, i.e. $X \times_k X$ Jacobson — which holds, being of finite type over $k = \bar{k}$. And that intersection is exactly the diagonal condition defining “variety” in the old sense. □

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Proof: The Local Criterion

Proof.

Suppose such an $\mathcal{O}$ existed. By Prop. 3, the local homomorphisms $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}$ and $\mathcal{O}_{X,y} \hookrightarrow \mathcal{O}$ define two $\mathcal{O}$ -valued points $f, g : \operatorname{Spec}(\mathcal{O}) \rightarrow X$ with $f(x_0) = x$, $g(x_0) = y$ at the closed point x_0 of $\operatorname{Spec}(\mathcal{O})$. At the generic point x_1 of $\operatorname{Spec}(\mathcal{O})$, both $f(x_1)$ and $g(x_1)$ equal the generic point of X , and $f_{x_1}^*, g_{x_1}^*$ both give the identity from $k(X)$ into the quotient field of $\mathcal{O}$. Hence

$$x_1 \in \{z : f(z) \equiv g(z)\}, \quad x_0 \notin \{z : f(z) \equiv g(z)\},$$

and since $x_0 \in \overline{\{x_1\}}$, this set is not closed — contradicting separatedness of X . □

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Proof: Intersections of Affines (Prop. 6)

Proof.

Let $\Delta : X \rightarrow X \times X$ be the diagonal, the product being absolute, i.e. over $\operatorname{Spec}(\mathbb{Z})$. (No generality is lost: X separated over an affine base is separated over $\mathbb{Z}$, since affine morphisms are separated and separatedness composes.) Since X is separated, $\Delta(X)$ is closed; covering X by open affines $U_i = \operatorname{Spec}(R_i)$, the diagonal is covered by the affines $U_i \times U_i$, on which Δ^* is the canonical surjection $R_i \otimes_{\mathbb{Z}} R_i \twoheadrightarrow R_i$. So Δ is a closed immersion. Now $U \times V$ is an open affine of $X \times X$ with ring $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V)$, hence $U \cap V = \Delta^{-1}(U \times V)$ is affine with ring a quotient of $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V)$. □

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